The Neutral Point in Stability and Control Analysis

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The neutral point is normally used to express static stability. If the center of gravity is ahead of the neutral point, the vehicle is statically stable, if aft, then statically unstable. This paper shows that the neutral point can also be used to simplify certain problems of stability and control. This is done by transferring the force and moment system to the neutral point of the configuration. The moment equation then is not a function of angle of attack and can be satisfied independently of the force equation. This allows for discussions of the effects of change of parameters directly from free-body diagrams of the force and moment system. The method is illustrated for the cases of trim, maneuverability, and stability of a vehicle.

Nomenclature

D = drag
L = lift
M = moment
R = turning radius
S = reference area
T = thrust
W = weight
V_0 = freestream (flight) velocity
NP = neutral point
MP = maneuver point
C_D = drag coefficient = $D/(1/2\rho V_0^2S)$
C_L = lift coefficient = $L/(1/2\rho V_0^2S)$
C_{L_α} = $\partial C_L/\partial\alpha$
C_{L_δ} = $\partial C_L/\partial\delta$
C_{L_q} = $\partial C_L/\partial q$
C_M = moment coefficient = $M/(1/2\rho V_0^2Sl)$
C_{M_0} = moment coefficient at zero lift
C_{M_α} = $\partial C_M/\partial\alpha$
C_{M_δ} = $\partial C_M/\partial\delta$
C_T = thrust coefficient = $T/(1/2\rho V_0^2S)$
h_n = distance from reference position to neutral point divided by l
h_m = distance from reference position to maneuver point divided by l
$h_{c.g.}$ = distance from reference position to center of gravity divided by l
i_B = dimensionless moment of inertia about pitching axis
l = reference length
l_c = distance between c.g. and control surface
l_c' = l_c/l
n = load factor = L/W
q = angular velocity in pitch
$\hat{q}$ = ql/V_0
α = angle of attack
$\dot{\alpha}$ = time rate of change of angle of attack
$\hat{\dot{\alpha}}$ = $\dot{\alpha}l/V_0$
δ = control surface deflection
μ = relative mass parameter = $W/\rho Slg$
ξ_n = stick-fixed static margin, i.e., the dimensionless distance between NP and c.g., positive when the NP is ahead of the c.g.
ξ_{-c} = stick-fixed static margin less the control force and moment
ξ_m = stick-fixed maneuver margin, i.e., the dimensionless distance between the NP and the maneuver point, MP, positive when the MP is ahead of the NP

Subscripts and Superscripts

$()'$ = moment is that about the neutral point
$-c$ = force or moment less that contributed by the control surface
c = force or moment due to the control surface

Introduction

THE method of approach presented in this paper is based on transferring the force and moment system to the neutral point of the configuration. The method was originated while the author was attempting to explain a peculiar behavior of an underwater missile, as predicted by the linearized equations of motion, to individuals not well versed in stability and control analysis. Since then, the author has adapted the approach in teaching an aeronautical engineering course in stability and control and in presenting a two-hour lecture on stability and control to a heterogeneous group of engineers during a short summer course in underwater missile engineering.

It is the author's opinion, based on the cited experience, that the method of approach presented in this paper brings out the effects of changes in the configuration parameters on the stability and controllability of a vehicle more clearly than the traditional method. This is especially true in trying to explain these effects to individuals not well versed in stability and control analysis.

The development of the linearized equations of motion and the study of the stability and control of airplane or missile-type configurations is adequately covered in the literature.[1-6] The neutral point is defined, in the pitch plane, as that point on the longitudinal axis of the configuration where the pitching moment does not vary with angle of attack. Its use is limited to static stability considerations. If the neutral point is aft of the c.g., the configuration is statically stable.

One can take advantage of the neutral point to formulate certain problems in a manner resulting in a simplification of the moment equation. This is illustrated for the following three cases: 1) the determination of level-flight trim conditions, 2) the determination of maneuverability, and 3) the stability of the short period mode.

Level Flight Trim

An airplane or missile in steady level flight is subjected to a force system shown acting at the c.g. in Fig. 1. The angle of attack α and the control surface deflection δ are considered to be small. The force system is dependent on the angle of attack and the control system deflection. The system is in equilibrium.

The force system of Fig. 1 is repeated in Fig. 2, with a linear dependence of the force and moment on the angle of attack and control deflection being assumed. The forces

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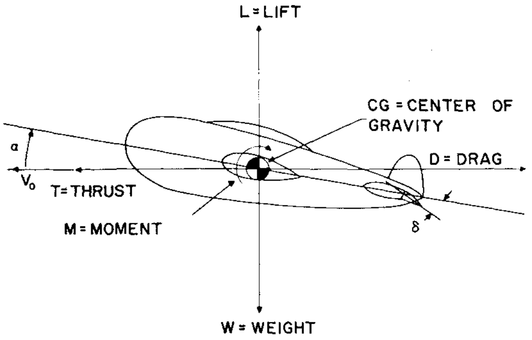

Fig. 1 Steady level flight.

and moment have been divided by $\frac{1}{2}\rho V_0^2 S$ and $\frac{1}{2}\rho V_0^2 Sl$, respectively, to place them in coefficient form, in which

ρ = density of the medium
V_0 = freestream velocity
S = a reference area
l = a reference length

Since the system is in equilibrium, the angle of attack and control surface deflection must be such that the lift coefficient is the trim lift coefficient, the moment coefficient is zero, and the thrust coefficient is equal to the drag coefficient at a given velocity and density. Another way of stating this is that the lift must equal the weight, the moment must be zero, and the thrust equal to the drag.

From Fig. 2, the two equations involving the two unknowns, the trim angle of attack and control surface deflection, can be written as

$$C_{L\alpha}\alpha_{\text{trim}} + C_{L\delta}\delta_{\text{trim}} = \frac{W}{\frac{1}{2}\rho V_0^2 S} = C_{L\text{trim}}$$
$$C_{M\alpha}\alpha_{\text{trim}} + C_{M\delta}\delta_{\text{trim}} + C_{M_0} = 0 \tag{1}$$

Equation (1) can be solved for the trim angle of attack and control surface deflection. The neutral point is defined as that point on the longitudinal axis of the body where the aerodynamic moment does not depend on angle of attack.† Transferring the equilibrium force system to the neutral point should therefore simplify the moment equation. The force system transferred to the neutral point is shown in Fig. 3.

The prime on the moment indicates that the moment is that about the neutral point. The neutral point is fixed for a given configuration at the distance $h_n l$ from some reference position. The term ξ_n is the static margin, that is, the nondimensional distance between the neutral point and the c.g. The thrust and drag forces are omitted from this figure.

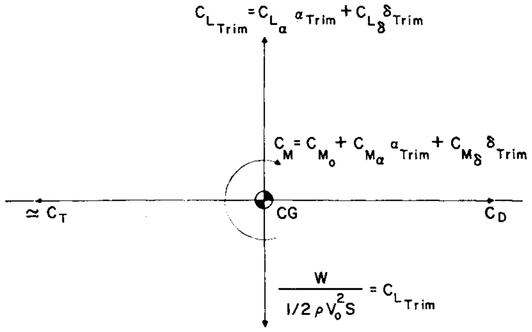

Fig. 2 Linearized coefficients for trim.

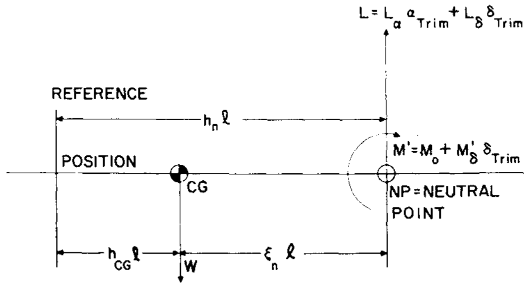

Fig. 3 Aerodynamic moments about the neutral point.

Figure 4 is the force system of Fig. 3 nondimensionalized. The moment equation, as pointed out previously and as seen in Figs. 3 or 4, does not now depend on the angle of attack. Compared with Fig. 2, Fig. 4 more clearly shows the effect of c.g. changes or changes in the configuration for a given configuration whose aerodynamic coefficients are linear with angle of attack and control surface deflection. There is no difference in the results of analysis based on either Fig. 2 or 4. Figure 4, however, can be discussed without writing down the equations (which is a handy device for use in discussions of qualitative data). This can be best illustrated by examples.

First, for a given velocity, altitude, and weight, consider the effects of c.g. variation on the trim angle of attack and control surface deflection. The moment due to the weight acting at a distance from the neutral point must be balanced by the moment due to the control surface deflection. The moment, and thus the control surface deflection, varies linearly with the distance of the c.g. from the neutral point ξ_n, the static margin. The moment coefficient due to the elevator angle will be positive for the elevators deflected downward. The variation of the control surface deflection δ with the static margin ξ_n is plotted on Fig. 5 as the solid line. The static margin ξ_n is positive for the c.g. ahead of the neutral point. The slope and intercepts of the line are easily determined from the moment relations shown in Fig. 4.

With δ_{trim} determined from the moment equation, the trim angle of attack α_{trim} is determined from the lift equation. The trim angle of attack varies linearly with the trim elevator deflection and therefore with the static margin. This variation is shown as the dotted line in Fig. 5. The slopes and intercepts are easily determined from the lift equation and the plot of the elevator deflection.

As a second example, consider the effect of the control surface location (at different longitudinal positions along the body) on the trim angles of attack and control surface deflection. For this example, let the control force be considered separate from the force of the rest of the configuration. In particular, let the subscript c on a force or moment indicate the force or moment due to the control surface. The subscript $-c$ on a force or moment will indicate the force or

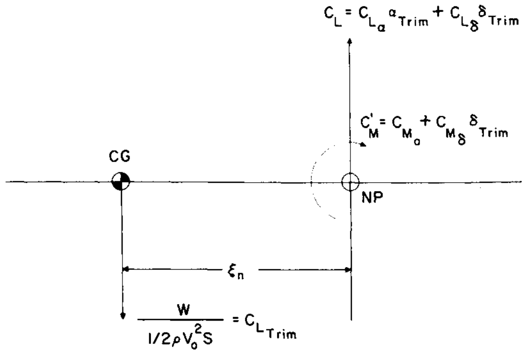

Fig. 4 Trim: aerodynamic relationships at the neutral point.

† The stick-fixed case only will be considered. The method can also be adapted to the stick-free case.

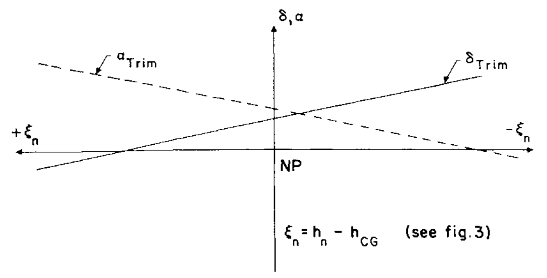

Fig. 5 Effect of c.g. location on trim angle of attack and control surface deflection.

moment of the configuration less that of the control surface. The lift and moment or the configuration are then

$$L = L_{-c} + L_c$$
$$M = M_{-c} + L_c l_c \tag{2}$$

where l_c is the distance between the center of application of the control force and the point about which the moment is written. The force and moment system is shown in Fig. 6.

The force and moment system less the control surface, transferred to the neutral point of the configuration less the control surface $(NP)_{-c}$, is shown in Fig. 7 in nondimensional form. The nondimensional distance between the stated neutral point, and the location of the control surface force is designated as l_c'.

The lift coefficient due to the control surface C_{L_c} is a function of angle of attack and control surface deflection. The value of the force of the control surface must be such as to balance the moment about $(NP)_{-c}$. For a given c.g. location, weight, velocity, and altitude, this moment is a constant. Assume that the unbalanced moment due to the weight and C_{M_0} about the neutral point is in the counterclockwise direction in Fig. 7. The control surface located aft of the neutral point would then have to exert a downward force. As the control surface approaches the neutral point, the force would have to increase. Since a constant control surface moment is required, the control force for trim varies inversely as the distance between the neutral point and the control surface force l_c'. When the control surface is forward of the neutral point, the direction of the control surface force has to be reversed. The control force plotted as a function of the control surface location along the body as measured from the neutral point is shown as the solid line in Fig. 8.

For the constant weight, c.g., altitude, and velocity, the sum of the lift due to the control surface and to the rest of the configuration must be a constant equal to the weight. This is shown in Fig. 8 in terms of the lift coefficients of Fig. 7. The required lift coefficient of the configuration less the control surface is shown as the dashed line. Since $C_{L_{-c}} = (C_{L\alpha})_{-c}\alpha$, the variation of α is also represented by the dashed line provided the vertical scale is changed accordingly. The linearizing assumption of small angles is violated when the control surface is near the neutral point. The actual force

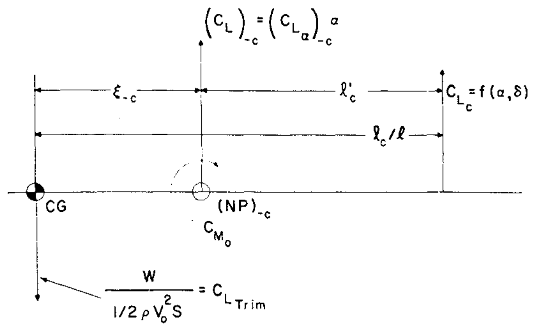

Fig. 7 Control surface location: aerodynamic relationships at the neutral point.

and moment system would therefore differ from that shown in Figs. 7 and 8 when the control surface is near the neutral point. Different curves are obtained if the neutral point is ahead of the c.g. or if the unbalanced moment is in the opposite direction.

Once the angle of attack α has been determined, the control surface deflection can be found for a given surface whose relationship $C_{L_c} = f(\alpha, \delta)$ with positions along the body is known.

If the neutral point of a configuration is located well forward, such as a slender body with small stabilizing surfaces, control surfaces located well forward could mean that the distance between the neutral point and the control surfaces would be rather small. The qualitative force system (or a similar one based on the unbalance moment opposite to the direction assumed for Fig. 8 or on the c.g. aft of the neutral point) of Fig. 8 shows that the forces required for trim, and consequently the trim angle of attack and control surface deflection, would be large. This is probably why configurations with forward control surfaces have very large stabilizing surfaces. The neutral points are quite far aft of the control surface location.

Maneuverability

The aerodynamic forces and moments during a maneuver are dependent on the pitching (or yawing) velocities, as well as the angle of attack and the control surface deflection. An indication of the maneuverability of the configuration is the elevator angle per gravity acceleration normal to the flight path required to maintain the vehicle in a steady turn. In the pitch plane, this value would change in a steady pull up because the weight component is fixed in direction, whereas the direction of the aerodynamic and propulsive forces depends on the position of the vehicle in the turn. The location in the loop considered for purposes of analysis is the bottom of the loop where the flight path is tangent to the horizontal.[2] Figure 9 is a sketch of the force system, in-

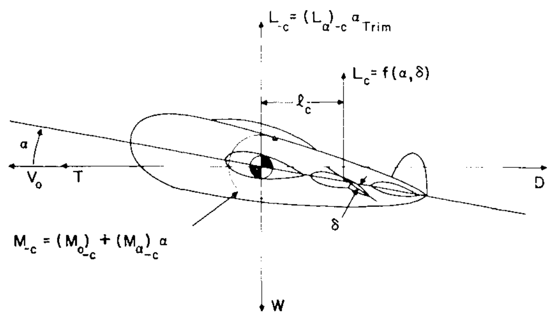

Fig. 6 Longitudinal variation of control force.

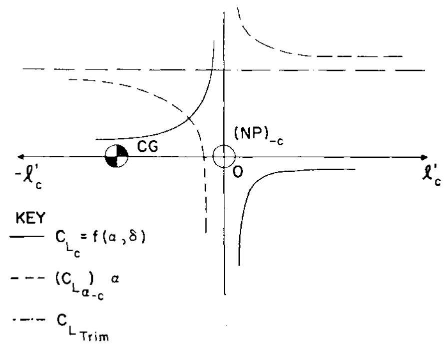

Fig. 8 Control force and angle-of-attack variation with control force location.

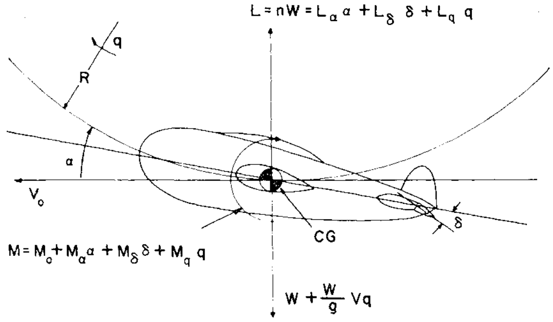

Fig. 9 Loop as an indication of maneuverability.

cluding a centrifugal force WVq acting on the vehicle at the bottom of the loop. The terms q and R are the angular velocity in pitch and turning radius, respectively. The lift being developed is nW, where n is the load factor.

Consider α and δ to be made up of two parts, the amount required for trim flight and the amount required for the maneuver. Then

$$L = L_\alpha(\alpha_{\text{trim}} + \alpha_{\text{man}}) + L_\delta(\delta_{\text{trim}} + \delta_{\text{man}}) + L_q q \quad \text{(3a)}$$

or

$$L = L_{\text{trim}} + L_{\text{man}} \quad \text{(3b)}$$

and

$$M = M_0 + M_\alpha(\alpha_{\text{trim}} + \alpha_{\text{man}}) + M_\delta(\delta_{\text{trim}} + \delta_{\text{man}}) + M_q q \quad \text{(3c)}$$

or

$$M = M_{\text{trim}} + M_{\text{man}} \quad \text{(3d)}$$

The condition shown in Fig. 9 is thus a sum of the two conditions shown in Fig. 10, the level flight trim at weight W, plus the "trim" to overcome the centrifugal force. The aerodynamic force and moment in the second condition shown in Fig. 10 also depend on the rotational velocity q.

The maneuverability can then be studied on the basis of the system shown in the right-hand sketch of Fig. 10. The left-hand side was investigated in the previous section of this paper.

The force and moment system in accordance with the procedure outlined in this paper is transferred to the neutral point. The nondimensional force system, equivalent to Fig. 4, is shown in Fig. 11.

The angular velocity with the cap over it $\hat{q}$ is a nondimensional angular velocity. The freestream velocity V_0 and a characteristic length l are used for the nondimensionalizing. The term μ is the nondimensional relative mass parameter.

For a constant angular velocity q, as well as a constant velocity and altitude, the control surface deflection necessary to maintain the loop is obtained from the moment relationship of Fig. 11. There is an unbalance of the moment due to the centrifugal force and of the moment due to the angular velocity (the damping moment) that is balanced by deflecting the control surface by δ_{man}. For the constant conditions stated, $C_{M\hat{q}}\hat{q}$ and $C_{L\hat{q}}\hat{q}$ are constant. The system of Fig. 11 is then similar to that of Fig. 4.

The moment and side force due to the angular velocity q vary with c.g. location. The variation of control surface deflection and angle of attack is, therefore, not linear with c.g. location as in Fig. 5. There is a location $\xi = \xi_m$ of the c.g. where the moment about the neutral point due to the centrifugal force just balances the damping moment due to the angular velocity q (Fig. 11). Since the damping moment is in the counterclockwise direction (that is, $C_{M\hat{q}}$ is negative) this point is located aft of the neutral point, as seen from Fig. 11. Whereas the neutral point depends on the geometry of the configuration only, the maneuver point depends on the weight and altitude as well. Its position from the neutral point, for $\delta_{\text{man}} = 0$, is given by the relationship

$$\xi = \xi_m = (1/2\mu)C_{M\hat{q}}' \quad \text{(4)}$$

determined from Fig. 11, where μ is the nondimensional relative mass parameter $W/\rho Slg$.

In terms of the neutral point, as measured from some reference point (Fig. 11),

$$h_m = h_n - \xi_m = h_n - (1/2\mu)C_{M\hat{q}}' \quad \text{(5)}$$

In order to be statically stable, the c.g. must be located forward of the neutral point. The maneuver point is aft of the c.g. In order to be as maneuverable as possible and still be statically stable, the c.g. should be forward of the neutral point and at as small a distance as possible.

Stick-Fixed Dynamic Stability

A simplified equation for the short period mode as given by the linearized equations of motion is obtained by letting the perturbation velocity in the x direction be zero and discarding the force equation in the same direction.[2] This results in the equation

$$\begin{aligned} (2\mu D - C_{L\hat{\dot{\alpha}}}D + C_{L\alpha})\alpha - (2\mu - C_{L\hat{q}})\hat{q} &= 0 \\ -(C_{M\hat{\dot{\alpha}}}D + C_{M\alpha})\alpha + (i_B D - C_{M\hat{q}})\hat{q} &= 0 \end{aligned} \quad \text{(6)}$$

where D is the differential operator d/dt and i_B is the nondimensional moment of inertia about the pitching axis.

The following roots of the characteristic equation of this system are obtained:

$$\lambda = \{-1 \pm [1 - (4AC/B^2)]^{1/2}\}\{B/2A\} \quad \text{(7)}$$

where

$$A = i_B(2\mu + C_{L\hat{\dot{\alpha}}})$$

$$B = i_B C_{L\alpha} - (2\mu + C_{L\hat{\dot{\alpha}}})C_{M\hat{q}} - (2\mu - C_{L\hat{q}})C_{M\hat{\dot{\alpha}}}$$

and

$$C = C_{M\hat{q}}C_{L\alpha} + C_{M\alpha}\,(2\mu - C_{L\hat{q}})$$

The terms $C_{L\hat{q}}$ and $C_{L\hat{\dot{\alpha}}}$ are frequently small compared to μ.

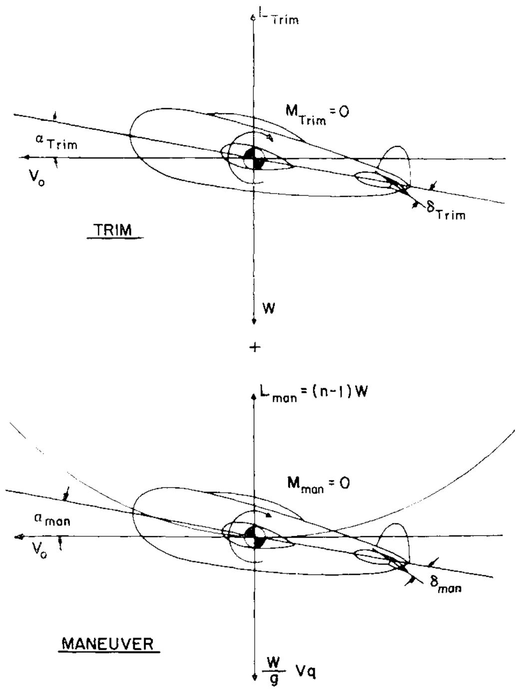

Fig. 10. Components of the loop.

Since μ, i_B, and $C_{L\alpha}$ are positive and $C_{M\hat{q}}$ and $C_{M\hat{\dot{\alpha}}}$ are negative, A and B are positive. The system will therefore be stable if the square root is imaginary or less than 1.0. Therefore, for stability,

$$C_{M\hat{q}}C_{L\alpha} + C_{M\alpha}(2\mu - C_{L\hat{q}}) < 0 \tag{8}$$

Thus, the fact whether the vehicle is stable or not (disregarding the type of transient) is dependent on the force and moment system due to the angle of attack of the body α and the pitching velocity $\hat{q}$.

The relationships of Eq. (8) obtained for stability suggest that the behavior of the vehicle at the bottom of a loop of constant radius will indicate whether or not it is stable (see Fig. 9). The force system of the right-hand side of Fig. 10 for the additional forces arising from the steady turn is repeated in Fig. 12. The value of the moment about the neutral point, noted as C_M' in Fig. 11, is now written in terms of the moment and force about the c.g. In addition, the moment due to the control surface is written in terms of the control force acting at some distance l_T from the c.g.

Consider the system of Fig. 12 in equilibrium prior to instantaneously zeroing the maneuver value of the control surface deflection. If the moment about the neutral point is negative, then making δ_{man} zero will cause the moment to bring the vehicle out of the loop. The relationship from Fig. 12 is

$$C_{M\hat{q}} + C_{L\hat{q}}\xi_n - 2\mu\xi_n < 0 \tag{9}$$

Since

$$\xi_n = \partial c_M/\partial C_L = C_{M\alpha}/C_{L\alpha} \tag{10}$$

for the neutral point ahead of the c.g., for the case shown in Fig. 12,

$$\xi_n = -(C_{M\alpha}/C_{L\alpha})$$

Therefore, substituting in Eq. (9),

$$C_{M\hat{q}}C_{L\alpha} - C_{L\hat{q}}C_{M\alpha} + 2\mu C_{M\alpha} < 0 \tag{11}$$

which is the relationship obtained in Eq. (8) from the linearized equations of motion as the condition for the dynamic stability of the vehicle.

If one is interested only in whether the vehicle is dynamically stable and not in the type of the transient, this criterion for dynamic stability can be expressed as in the following paragraph.

If the moment about the neutral point less the control force is in the direction tending to bring the vehicle out of the loop, the vehicle is dynamically stable. If it is in the direction tending to decrease the radius of the loop, the vehicle is dynamically unstable.

The force system of Fig. 12 illustrates the frequently mentioned criteria regarding torpedoes: 1) a stable torpedo will

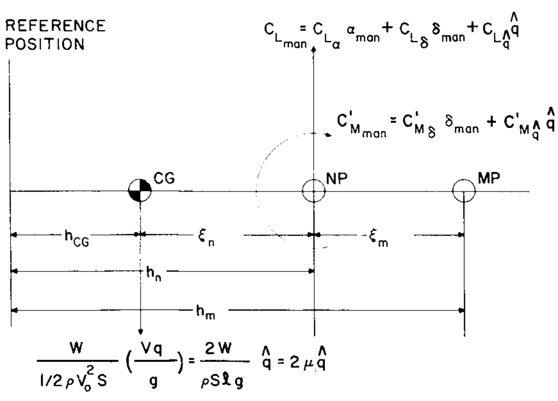

Fig. 11 Maneuver: aerodynamic relationships at the neutral point.

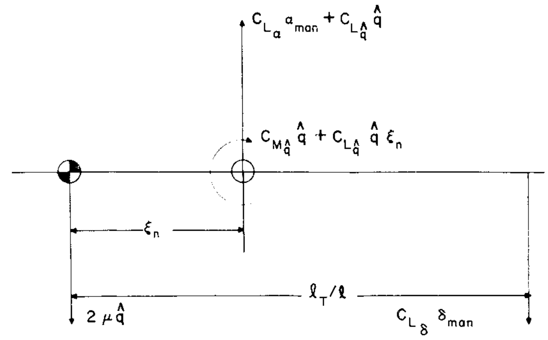

Fig. 12 Stability: aerodynamic relationships at the neutral point.

turn with its rudder (rudder moment in direction of turning) and an unstable one against its rudder; and 2) a stable torpedo in a turn will assume a straight-line course when the rudder is instantaneously reduced to zero, whereas an unstable one will tend to spiral into a tighter turn.[5]

The force system of Fig. 12 illustrates that a vehicle stable in water can become unstable on passing into air. If the moment about the neutral point less the control surface contribution is negative, the vehicle is stable. On passing through air, the density, and therefore the relative mass parameter, decreases by a factor of about 1000. This could result in the moment unbalance changing signs and the vehicle therefore becoming unstable.

Assume that the system of forces shown in Fig. 12 is that of a dynamically stable vehicle. The unbalanced moment (less the control moment) is therefore stable. Now, assume that the c.g. moves aft. At some point the unbalanced moment about the neutral point will become zero. For a c.g. location aft of this point, the unbalanced moment will change direction. Since the control surface deflection required for any turning rate is zero at this point, this location is the maneuver point discussed in a previous section of the paper. Thus, another way of stating the criteria for stability is that the c.g. must be ahead of the maneuver point.

Summary and Conclusions

The method of this report can be summarized as follows. The aerodynamic force and moment acting on a vehicle in steady level flight are functions of angle of attack and control surface deflection. The maneuverability and stability of the vehicle can be related to the condition of a steady turn (or loop) where, in addition to angle of attack and control surface deflection, the aerodynamic forces and moment are also functions of the turning rate.

The aerodynamic moment and force system is transferred to the neutral point, that is, to the point where the moment is independent of angle of attack. For trim considerations, the required control surface deflection is obtained directly from the moment equation. With the control surface deflection determined, the angle of attack is found from the force equation. For a given turning rate, the vehicle is more maneuverable the smaller the control surface deflection required to maintain the turning rate. The moment equation therefore is again independent of the force equation. Concerning stability (not the degree, but only whether stable or not), the vehicle is stable if the sum of the moment of the centrifugal force and the aerodynamic moment about the neutral point is in the direction to decrease the turning rate.

The method presented is not meant to take the place of the linearized equations of motion. It is only meant as a tool to supplement them. The author finds the method helpful in understanding the effects of configuration changes on trim, maneuverability, and stability.

Frequently the results of studies have to be presented to an individual not well versed in the linearized equations of motion. The method outlined has been found useful by the author in such circumstances, since it is based on free-body diagrams of the force system and is thus put into a form familiar to most engineers. The free-body system of this report differs from the ones frequently found in the literature in that the moments are considered about the neutral point rather than the c.g. This makes the moment independent of angle of attack and allows the discusser to point out qualitative effects of variations of parameters without writing any equations. He can satisfy the moment equation for control surface deflection independently of the force equation. Knowing the control surface deflection required, he can then determine the angle of attack required to satisfy the force equation. This is illustrated in the text for a number of cases. In a few places equations were written only for purposes of explaining a point in the paper. The equations obtained from the free-body diagrams are used to obtain numerical answers.

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